 2401

1) a) Integrating over $(-\pi,\pi)$ $\Rightarrow$ $\int_{-\pi}^{\pi} f(x)dx = a_0 . 2\pi + \sum 0$

Multiplying by $\cos mx$ $\Rightarrow$ $\int_{-\pi}^{\pi} \cos mx f(x)\,dx = 0 + a_m \int_{-\pi}^{\pi} \cos^2 mx\,dx + 0$

$= a_m . 2\pi \frac{1}{2}$

Similarly, multiplying by $\sin mx$. $\Rightarrow$ $b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx$

All use the orthogonality relations.

b) $\int_{-\pi}^{\pi}[f(x)-S_N(x)]^2dx = \int_{\pi}^{\pi} f(x)^2dx - 2\Big\{\sum_1^N a_n\int_{-\pi}^{\pi} f(x)\cos nx\,dx + b_n\int_{-\pi}^{\pi} f(x)\sin nx\,dx + a_0\int_{-\pi}^{\pi} f(x)\,dx\Big\}$

$+ \int_{-\pi}^{\pi}\Big\{a_0^2 + \sum_1^N a_n^2\cos^2 nx\,dx + \sum_1^N b_n^2 \sin^2 nx\,dx + 2a_0\big(\sum_1^N a_n\cos nx + b_n \sin nx\big) + 2\sum_{m\neq n}^N \sum_1^N a_m a_n \cos mx\cos nx + b_m b_n \sin mx \sin nx + 2\sum_{m=1}^N\sum_{n=1}^N a_n b_m \cos nx \sin mx\Big\}dx$

$\Rightarrow 0 \leq \int_{-\pi}^{\pi} f(x)^2dx - 2\Big\{\sum_1^N \pi a_n^2 + \pi b_n^2\big) + 2\pi a_0^2\Big\}$ using orthogonality relations

$+ \Big\{2\pi a_0^2 + \sum_1^N \pi a_n^2 + \pi b_n^2 + 0\Big\}$

$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2dx \geq 2a_0^2 + \sum_1^N a_n^2 + b_n^2.$

As series is bounded above it converges & so a_n & $b_n \to 0$

As $n\to\infty$ $S_n \to f$ & so equality is achieved

$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2dx = 2a_0^2 + \sum^{\infty} a_n^2 + b_n^2.$

d) $f(x)$ is odd so a_0 & $a_n = 0$ & $b_n = \frac{2}{\pi}\int_0^{\pi} x \sin$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx = \left[\frac{2}{\pi} x \frac{\cos nx}{n}(-1)\right]_0^{\pi} + \underbrace{\frac{2}{n\pi}\int_0^{\pi} \cos nx\,dx}_{=0}$$

$$= \frac{2(-1)^{n+1}}{n}$$

So $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \sum_1^{\infty} b_n^2 = \sum_1^{\infty} \frac{4}{n^2}$

ie $\sum_1^{\infty} \frac{1}{n^2} = \frac{2}{\pi}\cdot\frac{\pi^3}{3}\cdot\frac{1}{4} = \frac{\pi^2}{6}.$

$f = 3x^2 + 6xy + 2y^3 + 12x - 24y$

2 a) $f_x = 6x + 6y + 12$, $f_{xx} = 6$

$f_y = 6x + 6y^2 - 24$, $f_{yy} = 12y$ $\quad f_{xy} = 6$

Critical points have $f_x = f_y = 0 \Rightarrow x = -2-y$ & $x + y^2 = 4$

$\Rightarrow y^2 - y - 6 = 0$, $(y-3)(y+2) = 0$ $\quad y = 3, x = -5$ ①

$y = -2$ $x = 0$ ②

$\Delta = f_{xx}f_{yy} - f_{xy}^2 = 36(2y-1)$ >0 for ① where $f_{xx} > 0$

$\Rightarrow$ ① is a <u>minimum</u>

<0 for ② $\Rightarrow$ ② is a <u>saddle pt</u>

b) Consider $h(a,b,c) = f(a,b,c) - \lambda g(a,b,c)$ with

$f = A^2 = s(s-a)(s-b)(s-c)$ (s constant) $= 2L$

$g = (a+b+c) - L$ & set $h_a = h_b = h_c = 0$

So $-s(s-b)(s-c) = \lambda$, $-s(s-a)(s-c) = \lambda$, $-s(s-a)(s-b) = \lambda$

Dividing these gives, for example $\dfrac{s-b}{s-c} = 1 \Rightarrow b = c$

$\Rightarrow a = b = c$ by symmetry

$\Rightarrow a = b = c = L/3$

(This is trickier if $f = A$ is used)

(Also one may needlessly treat $s = \frac{1}{2}(a+b+c)$ in formula for f)

& not realise it can be treated as a constant.

3 a) E-L eqns are $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$

Consider $\dfrac{d}{dx}\left(F - y'\dfrac{\partial F}{\partial y'}\right) = \cancel{\dfrac{\partial F}{\partial x}}^{\,0} + y'\dfrac{\partial F}{\partial y} + \cancel{y''\dfrac{\partial F}{\partial y'}} - \cancel{y''\dfrac{\partial F}{\partial y'}}$
$- y'\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right)$

$= y'\left(\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right)\right) = 0 \Rightarrow \underline{F - y'\dfrac{\partial F}{\partial y'} = \text{Const}}$

b) $\displaystyle\int_1^2 x(y')^2 - yy'\,dx$, $\displaystyle\int_0^1 y\,dx = 1$, $y(1) = y(2) = 0$

Consider $\displaystyle\int_1^2 xy'^2 - yy' - \lambda y\,dx$ & the E.L. eqn gives

$-\cancel{y'} - \lambda = \dfrac{d}{dx}\left(2xy' - y\right) = \dfrac{d}{dx}\left(2xy'\right) - \cancel{y'}$

$\Rightarrow \dfrac{d}{dx}(xy') = \text{const } (= -\lambda/2) = A$ say

$xy' = Ax + B \Rightarrow y' = A + B/x$, $y = Ax + B\ln x + C$

& $y(1) = 0 \Rightarrow 0 = A + C \Rightarrow C = -A$
$y(2) = 0 \Rightarrow 0 = 2A + B\ln 2 + C = A + B\ln 2 \Rightarrow B = -A/\ln 2$

$y = A\left(x - \dfrac{\ln x}{\ln 2} - 1\right)$. A is chosen so constraint $\displaystyle\int_1^2 y\,dx = 1$

is satisfied $\Rightarrow 1/A = \displaystyle\int_1^2 x - \dfrac{\ln x}{\ln 2} - 1\,dx = \left[\dfrac{x^2}{2} - x - \dfrac{1}{\ln 2}(x\ln x - x)\right]_1^2$

$= \left(2 - 2 - \dfrac{1}{2} + 1\right) - \dfrac{1}{\ln 2}\left(2\ln 2 - 2 + 1\right) = \dfrac{1}{2} - 2 + \dfrac{1}{\ln 2} = \dfrac{1}{\ln 2} - \dfrac{3}{2}$

$\Rightarrow \underline{y = \dfrac{2\ln 2}{2 - 3\ln 2}\left\{x - \dfrac{\ln x}{\ln 2} - 1\right\}}$

4) a) $(3y-2u)u_x + (u-3x)u_y = (2x-y)$

χ eqns are $\frac{dx}{dt} = 3y-2u$, $\frac{dy}{dt} = u-3x$, $\frac{du}{dt} = 2x-y$

So on a χ, $\frac{d}{dt}(x^2+y^2+u^2) = 2x(3y-2u) + 2y(u-3x) + 2u(2x-y)$

$= 0 \Rightarrow x^2+y^2+u^2 = \text{const}$

& $\frac{d}{dt}(x+2y+3u) = (3y-2u) + 2(u-3x) + 3(2x-y) = 0$

$\Rightarrow x+2y+3u = \text{const}$

Lagrange's method now gives general solution as

$$x+2y+3u = f(x^2+y^2+u^2)$$

f is found from condition $u=0$ when $x=y \Rightarrow 3x = f(2x^2)$

& so $f(r) = 3\sqrt{r/2} \Rightarrow \underline{(x+2y+3u)^2 = (x^2+y^2+u^2)9/2}$

(similar seen)

$u_t + cu_x + \lambda u = 0$

b) Eqns of χ traces are $\frac{dt}{1} = \frac{dx}{c} \Rightarrow dx - c\,dt = 0$ $\quad x - ct = \text{const}$

Change variable from x & t to x & $s = x-ct$

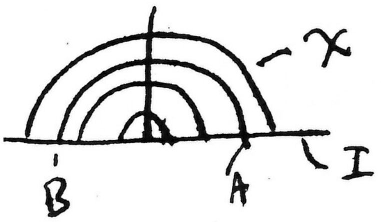
$\Rightarrow -c\frac{\partial u}{\partial s} + c\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial s}\right) = -\lambda u \Rightarrow u = f(s)e^{-\lambda x/c}$

$= f(x-ct)e^{-\lambda x/c}$

At $t=0$ $u(x,0) = F(x) \Rightarrow F(x) = f(x)e^{-\lambda x/c} \Rightarrow f(x) = F(x)e^{\lambda x/c}$

$u = F(x-ct)e^{\lambda(x-ct)/c}e^{-\lambda x/c} = \underline{e^{-\lambda t}F(x-ct)}$

c) Eqn χ traces of $yu_x - xu_y = c$ are $\frac{dx}{y} = \frac{dy}{-x} \Rightarrow x^2+y^2 = \text{const}$



χ are ½ circles centre $(0,0)$

The χ cross the line $y=0$, where Cauchy data is given, twice & unless c & f are suitable the data is inconsistent with the eqn.

A suitable f can be generated by defining ~~$f(x)$~~ only for $x>0$ & then defining f for $x<0$ consistent with soln found

5) a) $c^2 z_{xx} = z_{tt}$. Look for solutions $f(x+mt)$ & find $c^2 f'' = m^2 f'' \Rightarrow m = \pm c$. So general solution has the form

$$z = f(x+ct) + g(x-ct)$$
$$z_t = cf' - cg'$$

Initial conditions give $F = f+g$, $G = c(f'-g') \Rightarrow f'-g' = \frac{1}{c}\int_\alpha^x G(s)$

$\Rightarrow f(x) = \frac{1}{2}F(x) + \frac{1}{2c}\int_\alpha^x G(s)ds$, $g(x) = \frac{1}{2}F(x) - \frac{1}{2c}\int_\alpha^x G(s)ds$

& $z(x,t) = \frac{1}{2}\left(F(x+ct) + F(x-ct)\right) + \frac{1}{2c}\left(\int_\alpha^{x+ct} G(s)ds - \int_\alpha^{x-ct} G(s)ds\right)$

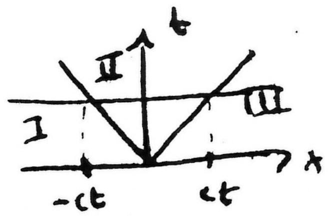
$$= \frac{1}{2}\left(F(x+ct) + F(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds$$

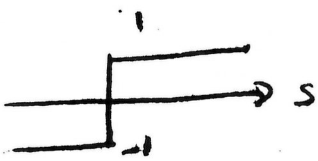
b) i) $z(x,0) = \frac{1}{2}\left(F(x+0) + F(x-0)\right) + \frac{1}{2c}\int_x^x G(s)ds$

$= F(x) + 0$ as required

ii) $z_t = \frac{1}{2}\left(cF'(x+ct) - cF'(x-ct)\right) + \frac{1}{2c}\left[G(x+ct)\underbrace{\frac{\partial}{\partial t}(x+ct)}_{=c} - G(x-ct)\underbrace{\frac{\partial}{\partial t}(x-ct)}_{=-c}\right]$

$z_t(x,0) = \frac{1}{2}c\left[F' - F'\right] + \frac{1}{2c}c\left[G+G\right] = G$ as required

c) 



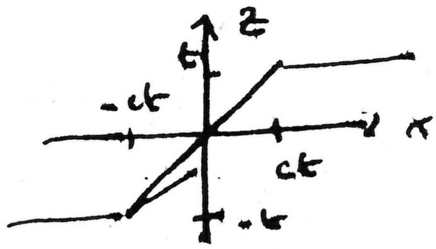
. In I $x-ct, x+ct < 0$ &

$z = \frac{1}{2c}\int_{x-ct}^{x+ct} -1\, ds = \frac{1}{2c}\left(-x-ct - (-x+ct)\right)$

In II, $x-ct < 0$, $x+ct > 0$, $z = \frac{1}{2c}\left\{\int_{x-ct}^{0} -1\, ds + \int_0^{x+ct} 1\, ds\right\} = \frac{1}{2c}\left(+x-ct + x+ct\right)$ $\underline{\underline{= -t}}$

$\underline{= \frac{x}{c}}$

In III, $x+ct > 0$, $x-ct > 0$, $z = \frac{1}{2c}\int_{x-ct}^{x+ct} 1\, ds = \underline{t}$



) (a) $\theta_{xx} = \theta_t$. Try $\theta(x,t) = X(x)T(t)$ & find $\frac{X''}{X} = \frac{T'}{T} = $ co

If (const) > 0 we have exponential solns in x & the homogeneous b.c. $\theta_x(0) = \theta(\pi) = 0$ cannot be satisfied

If (const) $= 0$, we have $X = Ax + B$ & again b.c. cannot be satisfied

If const $= -p^2 < 0$ we have $X(x) = A\cos px + B\sin px$. $\theta_x(0) \Rightarrow X'(0) = 0 \Rightarrow B = 0$ & then $\theta(\pi) = 0 \Rightarrow X(\pi) = 0 \Rightarrow \cos p\pi = 0$

$\Rightarrow p = (n + 1/2)$ $n = 0, 1, 2, \ldots$

Then $\frac{T'}{T} = -p^2 = -(n+1/2)^2$ & $T(t) = e^{-(n+1/2)^2 t}$

Solution is a sum of these normal modes so that

$$\theta(x,t) = \sum_0^\infty A_n e^{-(n+1/2)^2 t} \cos[(n+1/2)x]$$

If initially $\theta = 1$,

$$1 = \sum_0^\infty A_n \cos(n+1/2)x \Rightarrow \int_0^\pi \cos(n+1/2)x\,dx = A_n . \pi . \frac{1}{2}$$

$$\Rightarrow A_n = \frac{2}{\pi}\left[\frac{\sin(n+1/2)x}{(n+1/2)}\right]_0^\pi = \frac{2}{\pi}\frac{\sin(n+1/2)\pi}{n+1/2} = \frac{2}{\pi}\frac{(-1)^n}{(n+1/2)}$$

integrating $[\theta_x]_0^\pi = \int_0^\pi \frac{\partial\theta}{\partial t}dx \Rightarrow \theta_x(\pi) - \underbrace{\theta_x(0)}_{0} = \frac{d}{dt}\int_0^\pi \theta\,dx.$

$$\Rightarrow \frac{\partial}{\partial t}\int_0^\pi \theta\,dx = \sum_0^\infty A_n e^{-(n+1/2)^2 t} (-1) \sin(n+1/2)\pi$$

$$= -\frac{2}{\pi}\sum_0^\infty e^{-(n+1/2)^2 t}$$, which converges for $t > 0$

d) Integrating

$$\frac{\partial I}{\partial t} = -\frac{2}{\pi}\sum_0^\infty e^{-(n+1/2)^2 t} \Rightarrow I(t) = C + \frac{2}{\pi}\sum_0^\infty \frac{e^{-(n+1/2)^2 t}}{(n+1/2)^2}$$

At $t=0$ $\quad I(t) = \int_0^\pi 1\,dx = \pi \Rightarrow \pi = C + \frac{2}{\pi}\sum_0^\infty \frac{1}{(n+1/2)^2}$

As $t \to \infty$ $\quad I(t) \to 0$ as $\theta \to 0$ & so $C = 0$

$$\Rightarrow \quad \frac{\pi^2}{2} = \sum_0^\infty \frac{1}{(n+1/2)^2}.$$